

Infinite generation of non-cocompact lattices on right-angled buildings

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Let Γ be a non-cocompact lattice on a locally finite regular right-angled building X .
We prove that if Γ has a strict fundamental domain then Γ is not finitely generated.
We use the separation properties of subcomplexes of X called tree-walls.

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Tree lattices have been well-studied (see [BL]). Less understood are lattices on higher-dimensional CAT(0) complexes. In this paper, we consider lattices on X a locally finite, regular right-angled building (see Davis [D] and Section 1 below). Examples of such X include products of locally finite regular or biregular trees, or Bourdon's building $I_{p,q}$ [B], which has apartments hyperbolic planes tessellated by right-angled p -gons and all vertex links the complete bipartite graph $K_{q,q}$.

Let G be a closed, cocompact group of type-preserving automorphisms of X , equipped with the compact-open topology, and let Γ be a lattice in G . That is, Γ is discrete and the series $\sum |\text{Stab}_\Gamma(\phi)|^{-1}$ converges, where the sum is over the set of chambers ϕ of a fundamental domain for Γ . The lattice Γ is cocompact in G if and only if the quotient $\Gamma \backslash X$ is compact.

If there is a subcomplex $Y \subset X$ containing exactly one point from each Γ -orbit on X , then Y is called a *strict fundamental domain* for Γ . Equivalently, Γ has a strict fundamental domain if $\Gamma \backslash X$ may be embedded in X .

Any cocompact lattice in G is finitely generated. We prove:

Theorem 1 *Let Γ be a non-cocompact lattice in G . If Γ has a strict fundamental domain, then Γ is not finitely generated.*

We note that Theorem 1 contrasts with the finite generation of lattices on many buildings whose chambers are simplices. Results of, for example, Ballmann–Świątkowski [BS], Dymara–Januszkiewicz [DJ], and Zuk [Zu], establish that all lattices on many such buildings have Kazhdan's Property (T). Hence by a well-known result due to Kazhdan [K], these lattices are finitely generated.

Our proof of [Theorem 1](#), in [Section 3](#) below, uses the separation properties of sub-complexes of X which we call *tree-walls*. These generalize the tree-walls (in French, *arbre-murs*) of $I_{p,q}$, which were introduced by Bourdon in [\[B\]](#). We define tree-walls and establish their properties in [Section 2](#) below.

The following examples of non-cocompact lattices on right-angled buildings are known to us.

- (1) For $i = 1, 2$, let G_i be a rank one Lie group over a nonarchimedean locally compact field whose Bruhat–Tits building is the locally finite regular or biregular tree T_i . Then any irreducible lattice in $G = G_1 \times G_2$ is finitely generated (Raghunathan [\[Ra\]](#)). Hence by [Theorem 1](#) above, such lattices on $X = T_1 \times T_2$ cannot have strict fundamental domain.
- (2) Let Λ be a minimal Kac–Moody group over a finite field \mathbb{F}_q with right-angled Weyl group W . Then Λ has locally finite, regular right-angled twin buildings $X_+ \cong X_-$, and Λ acts diagonally on the product $X_+ \times X_-$. For q large enough:
 - (a) By Theorem 0.2 of Carbone–Garland [\[CG\]](#) or Theorem 1(i) of Rémy [\[Ré\]](#), the stabilizer in Λ of a point in X_- is a non-cocompact lattice in $\text{Aut}(X_+)$. Any such lattice is contained in a negative maximal spherical parabolic subgroup of Λ , which has strict fundamental domain a sector in X_+ , and so any such lattice has strict fundamental domain.
 - (b) By Theorem 1(ii) of Rémy [\[Ré\]](#), the group Λ is itself a non-cocompact lattice in $\text{Aut}(X_+) \times \text{Aut}(X_-)$. Since Λ is finitely generated, [Theorem 1](#) above implies that Λ does not have strict fundamental domain in $X = X_+ \times X_-$.
 - (c) By Section 7.3 of Gramlich–Horn–Mühlherr [\[GHM\]](#), the fixed set G_θ of certain involutions θ of Λ is a lattice in $\text{Aut}(X_+)$, which is sometimes cocompact and sometimes non-cocompact. Moreover, by [\[GHM, Remark 7.13\]](#), there exists θ such that G_θ is not finitely generated.
- (3) In [\[T\]](#), the first author constructed a functor from graphs of groups to complexes of groups, which extends the corresponding tree lattice to a lattice in $\text{Aut}(X)$ where X is a regular right-angled building. The resulting lattice in $\text{Aut}(X)$ has strict fundamental domain if and only if the original tree lattice has strict fundamental domain.

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1 Right-angled buildings

In this section we recall the basic definitions and some examples for right-angled buildings. We mostly follow Davis [D], in particular Section 12.2 and Example 18.1.10. See also [KT, Sections 1.2–1.4].

Let (W, S) be a right-angled Coxeter system. That is,

$$W = \langle S \mid (st)^{m_{st}} = 1 \rangle$$

where $m_{ss} = 1$ for all $s \in S$, and $m_{st} \in \{2, \infty\}$ for all $s, t \in S$ with $s \neq t$. We will discuss the following examples:

- $W_1 = \langle s, t \mid s^2 = t^2 = 1 \rangle \cong D_\infty$, the infinite dihedral group;
- $W_2 = \langle r, s, t \mid r^2 = s^2 = t^2 = (rs)^2 = 1 \rangle \cong (C_2 \times C_2) * C_2$, where C_2 is the cyclic group of order 2; and
- The Coxeter group W_3 generated by the set of reflections S in the sides of a right-angled hyperbolic p -gon, $p \geq 5$. That is,

$$W_3 = \langle s_1, \dots, s_p \mid s_i^2 = (s_i s_{i+1})^2 = 1 \rangle$$

with cyclic indexing.

Fix $(q_s)_{s \in S}$ a family of integers with $q_s \geq 2$. Given any family of groups $(H_s)_{s \in S}$ with $|H_s| = q_s$, let H be the quotient of the free product of the $(H_s)_{s \in S}$ by the normal subgroup generated by the commutators $\{[h_s, h_t] : h_s \in H_s, h_t \in H_t, m_{st} = 2\}$.

Now let X be the piecewise Euclidean CAT(0) geometric realization of the chamber system $\Phi = \Phi(H, \{1\}, (H_s)_{s \in S})$. Then X is a locally finite, regular right-angled building, with chamber set $\text{Ch}(X)$ in bijection with the elements of the group H . Let $\delta_W : \text{Ch}(X) \times \text{Ch}(X) \rightarrow W$ be the W -valued distance function and let $l_S : W \rightarrow \mathbb{N}$ be word length with respect to the generating set S . Denote by $d_W : \text{Ch}(X) \times \text{Ch}(X) \rightarrow \mathbb{N}$

the *gallery distance* $l_S \circ \delta_W$. That is, for two chambers ϕ and ϕ' of X , $d_W(\phi, \phi')$ is the length of a minimal gallery from ϕ to ϕ' .

Suppose that ϕ and ϕ' are s -adjacent chambers, for some $s \in S$. That is, $\delta_W(\phi, \phi') = s$. The intersection $\phi \cap \phi'$ is called an s -*panel*. By definition, since X is regular, each s -panel is contained in q_s distinct chambers. For distinct $s, t \in S$, the s -panel and t -panel of any chamber ϕ of X have nonempty intersection if and only if $m_{st} = 2$. Each s -panel of X is reduced to a vertex if and only if $m_{st} = \infty$ for all $t \in S - \{s\}$.

For the examples W_1 , W_2 , and W_3 above, respectively:

- The building X_1 is a tree with each chamber an edge, each s -panel a vertex of valence q_s , and each t -panel a vertex of valence q_t . That is, X_1 is the (q_s, q_t) -biregular tree. The apartments of X_1 are bi-infinite rays in this tree.
- The building X_2 has chambers and apartments as shown in [Figure 1](#) below. The r - and s -panels are 1-dimensional and the t -panels are vertices.

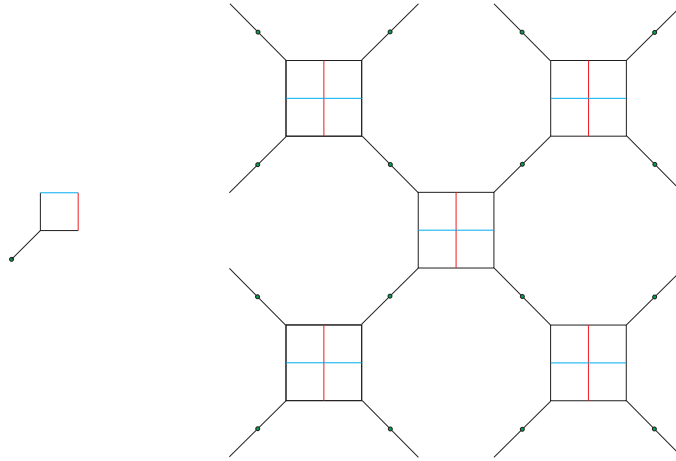


Figure 1: A chamber (on the left) and part of an apartment (on the right) for the building X_2 .

- The building X_3 has chambers p -gons and s -panels the edges of these p -gons. If $q_s = q \geq 2$ for all $s \in S$, then each s -panel is contained in q chambers, and X_3 , equipped with the obvious piecewise hyperbolic metric, is Bourdon's building $I_{p,q}$.

2 Tree-walls

We now generalize the notion of tree-wall due to Bourdon [B]. We will use basic facts about buildings, found in, for example, Davis [D]. Our main results concerning tree-walls are [Corollary 3](#) below, which describes three possibilities for tree-walls, and [Proposition 6](#) below, which generalizes the separation property 2.4.A(ii) of [B].

Let X be as in [Section 1](#) above and let $s \in S$. As in [B, Section 2.4.A], we define two s -panels of X to be *equivalent* if they are contained in a common wall of type s in some apartment of X . A *tree-wall of type s* is then an equivalence class under this relation. We note that in order for walls and thus tree-walls to have a well-defined type, it is necessary only that all finite m_{st} , for $s \neq t$, be even. Tree-walls could thus be defined for buildings of type any even Coxeter system, and they would have properties similar to those below. We will however only explicitly consider the right-angled case.

Let \mathcal{T} be a tree-wall of X , of type s . We define a chamber ϕ of X to be *epicormic at \mathcal{T}* if the s -panel of ϕ is contained in \mathcal{T} , and we say that a gallery $\alpha = (\phi_0, \dots, \phi_n)$ *crosses \mathcal{T}* if, for some $0 \leq i < n$, the chambers ϕ_i and ϕ_{i+1} are epicormic at \mathcal{T} .

By the definition of tree-wall, if $\phi \in \text{Ch}(X)$ is epicormic at \mathcal{T} and $\phi' \in \text{Ch}(X)$ is t -adjacent to ϕ with $t \neq s$, then ϕ' is epicormic at \mathcal{T} if and only if $m_{st} = 2$. Let $s^\perp := \{t \in S \mid m_{st} = 2\}$ and denote by $\langle s^\perp \rangle$ the subgroup of W generated by the elements of s^\perp . If s^\perp is empty then by convention, $\langle s^\perp \rangle$ is trivial. For the examples in [Section 1](#) above:

- in W_1 , both $\langle s^\perp \rangle$ and $\langle t^\perp \rangle$ are trivial;
- in W_2 , $\langle r^\perp \rangle = \langle s \rangle \cong C_2$ and $\langle s^\perp \rangle = \langle r \rangle \cong C_2$, while $\langle t^\perp \rangle$ is trivial; and
- in W_3 , $\langle s_i^\perp \rangle = \langle s_{i-1}, s_{i+1} \rangle \cong D_\infty$ for each $1 \leq i \leq p$.

Lemma 2 *Let \mathcal{T} be a tree-wall of X of type s . Let ϕ be a chamber which is epicormic at \mathcal{T} and let A be any apartment containing ϕ .*

- (1) *The intersection $\mathcal{T} \cap A$ is a wall of A , hence separates A .*
- (2) *There is a bijection between the elements of the group $\langle s^\perp \rangle$ and the set of chambers of A which are epicormic at \mathcal{T} and in the same component of $A - \mathcal{T} \cap A$ as ϕ .*

Proof Part (1) is immediate from the definition of tree-wall. For Part (2), let $w \in \langle s^\perp \rangle$ and let $\psi = \psi_w$ be the unique chamber of A such that $\delta_w(\phi, \psi) = w$. We claim that ψ is epicormic at \mathcal{T} and in the same component of $A - \mathcal{T} \cap A$ as ϕ .

For this, let $s_1 \cdots s_n$ be a reduced expression for w and let $\alpha = (\phi_0, \dots, \phi_n)$ be the minimal gallery from $\phi = \phi_0$ to $\psi = \phi_n$ of type (s_1, \dots, s_n) . Since w is in $\langle s^\perp \rangle$, we have $m_{s_i s} = 2$ for $1 \leq i \leq n$. Hence by induction each ϕ_i is epicormic at \mathcal{T} , and so $\psi = \phi_n$ is epicormic at \mathcal{T} . Moreover, since none of the s_i are equal to s , the gallery α does not cross \mathcal{T} . Thus $\psi = \psi_w$ is in the same component of $A - \mathcal{T} \cap A$ as ϕ .

It follows that $w \mapsto \psi_w$ is a well-defined, injective map from $\langle s^\perp \rangle$ to the set of chambers of A which are epicormic at \mathcal{T} and in the same component of $A - \mathcal{T} \cap A$ as ϕ . To complete the proof, we will show that this map is surjective. So let ψ be a chamber of A which is epicormic at \mathcal{T} and in the same component of $A - \mathcal{T} \cap A$ as ϕ , and let $w = \delta_W(\phi, \psi)$.

If $\langle s^\perp \rangle$ is trivial then $\psi = \phi$ and $w = 1$, and we are done. Next suppose that the chambers ϕ and ψ are t -adjacent, for some $t \in S$. Since both ϕ and ψ are epicormic at \mathcal{T} , either $t = s$ or $m_{st} = 2$. But ψ is in the same component of $A - \mathcal{T} \cap A$ as ϕ , so $t \neq s$, hence $w = t$ is in $\langle s^\perp \rangle$ as required. If $\langle s^\perp \rangle$ is finite, then finitely many applications of this argument will finish the proof. If $\langle s^\perp \rangle$ is infinite, we have established the base case of an induction on $n = l_S(w)$.

For the inductive step, let $s_1 \cdots s_n$ be a reduced expression for w and let $\alpha = (\phi_0, \dots, \phi_n)$ be the minimal gallery from $\phi = \phi_0$ to $\psi = \phi_n$ of type (s_1, \dots, s_n) . Since ϕ and ψ are in the same component of $A - \mathcal{T} \cap A$ and α is minimal, the gallery α does not cross \mathcal{T} . We claim that s_n is in s^\perp . First note that $s_n \neq s$ since α does not cross \mathcal{T} and $\psi = \phi_n$ is epicormic at \mathcal{T} . Now denote by \mathcal{T}_n the tree-wall of X containing the s_n -panel $\phi_{n-1} \cap \phi_n$. Since α is minimal and crosses \mathcal{T}_n , the chambers $\phi = \phi_0$ and $\psi = \phi_n$ are separated by the wall $\mathcal{T}_n \cap A$. Thus the s -panel of ϕ and the s -panel of ψ are separated by $\mathcal{T}_n \cap A$. As the s -panels of both ϕ and ψ are in the wall $\mathcal{T} \cap A$, it follows that the walls $\mathcal{T}_n \cap A$ and $\mathcal{T} \cap A$ intersect. Hence $m_{s_n s} = 2$, as claimed.

Now let $w' = ws_n = s_1 \cdots s_{n-1}$ and let ψ' be the unique chamber of A such that $\delta_W(\phi, \psi') = w'$. Since s_n is in s^\perp and ψ' is s_n -adjacent to ψ , the chamber ψ' is epicormic at \mathcal{T} and in the same component of $A - \mathcal{T} \cap A$ as ϕ . Moreover $s_1 \cdots s_{n-1}$ is a reduced expression for w' , so $l_S(w') = n - 1$. Hence by the inductive assumption, w' is in $\langle s^\perp \rangle$. Therefore $w = w's_n$ is in $\langle s^\perp \rangle$, which completes the proof. \square

Corollary 3 *The following possibilities for tree-walls in X may occur.*

- (1) *Every tree-wall of type s is reduced to a vertex if and only if $\langle s^\perp \rangle$ is trivial.*
- (2) *Every tree-wall of type s is finite but not reduced to a vertex if and only if $\langle s^\perp \rangle$ is finite but nontrivial.*

(3) Every tree-wall of type s is infinite if and only if $\langle s^\perp \rangle$ is infinite.

Proof Let \mathcal{T} , ϕ , and A be as in Lemma 2 above. The set of s -panels in the wall $\mathcal{T} \cap A$ is in bijection with the set of chambers of A which are epicormic at \mathcal{T} and in the same component of $A - \mathcal{T} \cap A$ as ϕ . \square

For the examples in Section 1 above:

- in X_1 , every tree-wall of type s and of type t is a vertex;
- in X_2 , the tree-walls of types both r and s are finite and 1-dimensional, while every tree-wall of type t is a vertex; and
- in X_3 , all tree-walls are infinite, and are 1-dimensional.

Corollary 4 Let \mathcal{T} , ϕ , and A be as in Lemma 2 above and let

$$\rho = \rho_{\phi, A}: X \rightarrow A$$

be the retraction onto A centered at ϕ . Then $\rho^{-1}(\mathcal{T} \cap A) = \mathcal{T}$.

Proof Let ψ be any chamber of A which is epicormic at \mathcal{T} and is in the same component of $A - \mathcal{T} \cap A$ as ϕ . Then by the proof of Lemma 2 above, $w := \delta_W(\phi, \psi)$ is in $\langle s^\perp \rangle$. Let ψ' be a chamber in the preimage $\rho^{-1}(\psi)$ and let A' be an apartment containing both ϕ and ψ' . Since the retraction ρ preserves W -distances from ϕ , we have that $\delta_W(\phi, \psi') = w$ is in $\langle s^\perp \rangle$. Again by the proof of Lemma 2, it follows that the chamber ψ' is epicormic at \mathcal{T} . But the image under ρ of the s -panel of ψ' is the s -panel of ψ . Thus $\rho^{-1}(\mathcal{T} \cap A) = \mathcal{T}$, as required. \square

Lemma 5 Let \mathcal{T} be a tree-wall and let ϕ and ϕ' be two chambers of X . Let α be a minimal gallery from ϕ to ϕ' and let β be any gallery from ϕ to ϕ' . If α crosses \mathcal{T} then β crosses \mathcal{T} .

Proof Suppose that α crosses \mathcal{T} . Since α is minimal, there is an apartment A of X which contains α , and hence the wall $\mathcal{T} \cap A$ separates ϕ from ϕ' . Choose a chamber ϕ_0 of A which is epicormic at \mathcal{T} and consider the retraction $\rho = \rho_{\phi_0, A}$ onto A centered at ϕ_0 . Since ϕ and ϕ' are in A , ρ fixes ϕ and ϕ' . Hence $\rho(\beta)$ is a gallery in A from ϕ to ϕ' , and so $\rho(\beta)$ crosses $\mathcal{T} \cap A$. By Corollary 4 above, $\rho^{-1}(\mathcal{T} \cap A) = \mathcal{T}$. Therefore β crosses \mathcal{T} . \square

Proposition 6 Let \mathcal{T} be a tree-wall of type s . Then \mathcal{T} separates X into q_s gallery-connected components.

Proof Fix an s -panel in \mathcal{T} and let $\phi_1, \dots, \phi_{q_s}$ be the q_s chambers containing this panel. Then for all $1 \leq i < j \leq q_s$, the minimal gallery from ϕ_i to ϕ_j is just (ϕ_i, ϕ_j) , and hence crosses \mathcal{T} . Thus by [Lemma 5](#) above, any gallery from ϕ_i to ϕ_j crosses \mathcal{T} . So the q_s chambers $\phi_1, \dots, \phi_{q_s}$ lie in q_s distinct components of $X - \mathcal{T}$.

To complete the proof, we show that \mathcal{T} separates X into at most q_s components. Let ϕ be any chamber of X . Then among the chambers $\phi_1, \dots, \phi_{q_s}$, there is a unique chamber, say ϕ_1 , at minimal gallery distance from ϕ . It suffices to show that ϕ and ϕ_1 are in the same component of $X - \mathcal{T}$.

Let α be a minimal gallery from ϕ to ϕ_1 and let A be an apartment containing α . Then there is a unique chamber of A which is s -adjacent to ϕ_1 . Hence A contains ϕ_i for some $i > 1$, and the wall $\mathcal{T} \cap A$ separates ϕ_1 from ϕ_i . Since α is minimal and $d_W(\phi, \phi_1) < d_W(\phi, \phi_i)$, the Exchange Condition (see [\[D, page 35\]](#)) implies that a minimal gallery from ϕ to ϕ_i may be obtained by concatenating α with the gallery (ϕ_1, ϕ_i) . Since a minimal gallery can cross $\mathcal{T} \cap A$ at most once, α does not cross $\mathcal{T} \cap A$. Thus ϕ and ϕ_1 are in the same component of $X - \mathcal{T}$, as required. \square

3 Proof of Theorem

Let G be as in the introduction and let Γ be a non-cocompact lattice in G with strict fundamental domain. Fix a chamber ϕ_0 of X . For each integer $n \geq 0$ define

$$D(n) := \{ \phi \in \text{Ch}(X) \mid d_W(\phi, \Gamma\phi_0) \leq n \}.$$

Then $D(0) = \Gamma\phi_0$, and for every $n > 0$ every connected component of $D(n)$ contains a chamber in $\Gamma\phi_0$. To prove [Theorem 1](#), we will show that there is no $n > 0$ such that $D(n)$ is connected.

Let Y be a strict fundamental domain for Γ which contains ϕ_0 . For each chamber ϕ of X , denote by ϕ_Y the representative of ϕ in Y .

Lemma 7 *Let ϕ and ϕ' be t -adjacent chambers in X , for $t \in S$. Then either $\phi_Y = \phi'_Y$, or ϕ_Y and ϕ'_Y are t -adjacent.*

Proof It suffices to show that the t -panel of ϕ_Y is the t -panel of ϕ'_Y . Since Y is a subcomplex of X , the t -panel of ϕ_Y is contained in Y . By definition of a strict fundamental domain, there is exactly one representative in Y of the t -panel of ϕ . Hence the unique representative in Y of the t -panel of ϕ is the t -panel of ϕ_Y . Similarly, the unique representative in Y of the t -panel of ϕ' is the t -panel of ϕ'_Y . But ϕ and ϕ' are

t -adjacent, hence have the same t -panel, and so it follows that ϕ_Y and ϕ'_Y have the same t -panel. \square

Corollary 8 *The fundamental domain Y is gallery-connected.*

Lemma 9 *For all $n > 0$, the fundamental domain Y contains a pair of adjacent chambers ϕ_n and ϕ'_n such that, if \mathcal{T}_n denotes the tree-wall separating ϕ_n from ϕ'_n :*

- (1) *the chambers ϕ_0 and ϕ_n are in the same gallery-connected component of $Y - \mathcal{T}_n \cap Y$;*
- (2) *$\min\{d_W(\phi_0, \phi) \mid \phi \in \text{Ch}(X) \text{ is epicormic at } \mathcal{T}_n\} > n$; and*
- (3) *there is a $\gamma \in \text{Stab}_\Gamma(\phi'_n)$ which does not fix ϕ_n .*

Proof Fix $n > 0$. Since Γ is not cocompact, Y is not compact. Thus there exists a tree-wall \mathcal{T}_n with $\mathcal{T}_n \cap Y$ nonempty such that for every $\phi \in \text{Ch}(X)$ which is epicormic at \mathcal{T}_n , $d_W(\phi_0, \phi) > n$. Let s_n be the type of the tree-wall \mathcal{T}_n . Then by [Corollary 8](#) above, there is a chamber ϕ_n of Y which is epicormic at \mathcal{T}_n and in the same gallery-connected component of $Y - \mathcal{T}_n \cap Y$ as ϕ_0 , such that for some chamber ϕ'_n which is s_n -adjacent to ϕ_n , ϕ'_n is also in Y . Now, as Γ is a non-cocompact lattice, the orders of the Γ -stabilizers of the chambers in Y are unbounded. Hence the tree-wall \mathcal{T}_n and chambers ϕ_n and ϕ'_n may be chosen so that $|\text{Stab}_\Gamma(\phi_n)| < |\text{Stab}_\Gamma(\phi'_n)|$. \square

Let ϕ_n , ϕ'_n , \mathcal{T}_n , and γ be as in [Lemma 9](#) above and let $s = s_n$ be the type of the tree-wall \mathcal{T}_n . Let α be a gallery in $Y - \mathcal{T}_n \cap Y$ from ϕ_0 to ϕ_n . The chambers ϕ_n and $\gamma \cdot \phi_n$ are in two distinct components of $X - \mathcal{T}_n$, since they both contain the s -panel $\phi_n \cap \phi'_n \subseteq \mathcal{T}_n$, which is fixed by γ . Hence the galleries α and $\gamma \cdot \alpha$ are in two distinct components of $X - \mathcal{T}_n$, and so the chambers ϕ_0 and $\gamma \cdot \phi_0$ are in two distinct components of $X - \mathcal{T}_n$. Denote by X_0 the component of $X - \mathcal{T}_n$ which contains ϕ_0 , and put $Y_0 = Y \cap X_0$.

Lemma 10 *Let ϕ be a chamber in X_0 that is epicormic at \mathcal{T}_n . Then ϕ_Y is in Y_0 and is epicormic at $\mathcal{T}_n \cap Y$.*

Proof We consider three cases, corresponding to the possibilities for tree-walls in [Corollary 3](#) above.

- (1) If \mathcal{T}_n is reduced to a vertex, there is only one chamber in X_0 which is epicormic at \mathcal{T}_n , namely ϕ_n . Thus $\phi = \phi_n = \phi_Y$ and we are done.

- (2) If \mathcal{T}_n is finite but not reduced to a vertex, the result follows by finitely many applications of [Lemma 7](#) above.
- (3) If \mathcal{T}_n is infinite, the result follows by induction, using [Lemma 7](#) above, on

$$k := \min\{d_W(\phi, \psi) \mid \psi \text{ is a chamber of } Y_0 \text{ epicormic at } \mathcal{T}_n \cap Y\}. \quad \square$$

Lemma 11 *For all $n > 0$, the complex $D(n)$ is not connected.*

Proof Fix $n > 0$, and let α be a gallery in X between a chamber in $X_0 \cap \Gamma\phi_0$ and some chamber ϕ in X_0 that is epicormic at \mathcal{T}_n . Let m be the length of α .

By [Lemma 7](#) and [Lemma 10](#) above, the gallery α projects to a gallery β in Y between ϕ_0 and a chamber ϕ_Y that is epicormic at $\mathcal{T}_n \cap Y$. The gallery β in Y has length at most m .

It follows from (2) of [Lemma 9](#) above that the gallery β in Y has length greater than n . Therefore $m > n$. Hence the gallery-connected component of $D(n)$ that contains ϕ_0 is contained in X_0 . As the chamber $\gamma \cdot \phi_0$ is not in X_0 , it follows that the complex $D(n)$ is not connected. \square

This completes the proof, as Γ is finitely generated if and only if $D(n)$ is connected for some n .

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